ON A CLASS OF MULTIVALENT FUNCTIONS DEFINED BY GENERALIZED RUSCHEWEYH DERIVATIVES ASSOCIATED WITH FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT

By making use of generalized Ruscheweyh derivative operator we define a new class $N_p(\lambda, \eta, \gamma)$ of multivalent functions associated with fractional derivative operator. We have derived some results on coefficient estimate, closure theorem, radius of starlikeness and convexity. Also we have applied general fractional derivative operator techniques to obtain some results for the functions belonging to this class.

Keywords

Coefficient estimate, Radius of starlikeness, Radius of convexity, Hadamard product, Generalized Ruscheweyh derivatives, Closure theorem, Generalized fractional derivative operator.

1. INTRODUCTION

Let A(p) denote the class of analytic and p-valent functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k z^k \tag{1}$$

in the open unit disc $U = \{z; z \in \mathbb{C}, |z| < 1\}$. Where p is some positive integer.

Let f(z) is given by (1) and g(z) is given by

$$g(z) = z^p - \sum_{k=n+p}^{\infty} \beta_k z^k \tag{2}$$

We define convolution product (or hadamard product) of f and g by

$$(f * g)(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k \beta_k z^k \tag{3}$$

Generalized fractional derivative operator of order γ is defined by Srivastava and Silverman [1,2]

$$J_{0,z}^{\gamma,\mu,\vartheta} f(z) = \begin{cases} \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \left\{ z^{\gamma-\mu} \int_0^z (z-t)^{-\gamma} \; {}_2F_1\left(\mu-\gamma, 1-\vartheta; 1-\gamma, 1-\frac{t}{z}\right\} f(t)dt, & (0 \le \gamma < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\gamma-n,\mu,\vartheta} f(z), & (n \le \gamma < n+1, n \in N) \end{cases} \tag{4}$$

Where f is an analytic function in a simply connected region of complex z-plane containing the origin, and the multiplicity of (z-t)-γ is removed by requiring log (z-t) to be real when z-t > 0. In term of gamma function, we have

$$J_{0,z}^{\gamma,\mu,\vartheta} z^p = \frac{\Gamma(p+1)\Gamma(p-\mu+\vartheta+2)}{\Gamma(p-\mu+1)\Gamma(p-\gamma+\vartheta+2)} z^{p-\mu}, \; (0 \le \gamma < 1, p > \max\{0, \mu-\vartheta-1\} - 1) \tag{5}$$

From the above definition we have $D_z^{\gamma} f(z)$ fractional derivative operator of order γ.

$$J_{0,z}^{\gamma,\vartheta} f(z) = D_z^{\gamma} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\gamma}} dt, \quad (0 \le \gamma < 1) \tag{6}$$

In terms of gamma function we have

$$D_z^{\gamma} z^p = \frac{\Gamma(p+1)}{\Gamma(p-\gamma+1)} z^{p-\gamma}, \quad (0 \le \gamma) \tag{7}$$

Fractional integral operator of order γ is also given by

$$D_z^{-\gamma} f(z) = \frac{1}{\Gamma(\gamma)} \int_0^z \frac{f(t)}{(z-t)^{1-\gamma}} dt, \tag{8}$$

Where f(z) is an analytic function in a simply-connected region of the complex z-plane containing the origin and the multiplicity of (z-t)γ-1 is removed by requiring log(z-t) to the real when z-t > 0. In terms of gamma function, we have

$$D_z^{-\gamma} z^p = \frac{\Gamma(p+1)}{\Gamma(p+\gamma+1)} z^{p+\gamma} \tag{9}$$

Goyal [2] defined generalized Ruscheweyh derivative as follows:

$$J_p^{\gamma,\mu} f(z) = \frac{\Gamma(\mu-\gamma++2)}{\Gamma(\vartheta+2)\Gamma(\mu+1)} z^p J_{0,z}^{\gamma,\mu,\vartheta}(z^{\mu-p} f(z)) = z^p - \sum_{k=n+p}^{\infty} \alpha_k A_p^{\gamma,\mu}(k) z^k \tag{10}$$

where

$$A_p^{\gamma,\mu}(k) = \frac{\Gamma(k-p+1)\Gamma(\vartheta+2\mu-\gamma)\Gamma(k+\vartheta-p+2)}{\Gamma(k-p+1)\Gamma(k+\vartheta-p+2+\mu-\gamma)\Gamma(\vartheta+2)\Gamma(1+\mu)} \tag{11}$$

For γ = μ generalized Ruscheweyh derivatives get reduced to Ruscheweyh derivatives of f(z) of order γ [4]

$$D^{\gamma} f(z) = \frac{z^p}{\Gamma(\gamma+1)} \frac{d^{\gamma}}{dz^{\gamma}} \left(z^{\gamma-p} f(z)\right) = z^p + \sum_{k=n+p}^{\infty} \alpha_k A_k(\gamma) z^k \tag{12}$$

where

$$A_k(\gamma) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma+p)\gamma(k-p+1)} \tag{13}$$

Definition:- A function is said to be in the class $N_p(\lambda, \eta, \gamma)$, if it satisfies the condition

$$\mathrm{Re}\left\{\frac{z\left(J_p^{\gamma,\mu} f(z)\right)' + \lambda z^2 \left(J_p^{\gamma,\mu} f(z)\right)''}{(1-\lambda)\left(J_p^{\gamma,\mu} f(z)\right) + \lambda z \left(J_p^{\gamma,\mu} f(z)\right)'}\right\} > \eta \tag{14}$$

for $z \in U$ and $0 \le \lambda < 1, 0 \le \eta < p$ *and* $\gamma > -1$.

2. COEFFICIENT ESTIMATE

Theorem1:- Let $f(z) \in A(p)$, $z \in U$ be of the form (1). Then $f(z) \in N_p(\lambda, \eta, \gamma)$ if

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} \alpha_k A_p^{\gamma,\mu}(k) < 1 \tag{15}$$

where $0 \le \lambda < 1, 0 \le \eta < p$ *and* $\gamma > -1$.

Proof: Suppose that $f(z) \in N_p(\lambda, \eta, \gamma)$ then

$$\mathrm{Re}\left\{\frac{z\left(J_p^{\gamma,\mu} f(z)\right)' + \lambda z^2 \left(J_p^{\gamma,\mu} f(z)\right)''}{(1-\lambda)\left(J_p^{\gamma,\mu} f(z)\right) + \lambda z \left(J_p^{\gamma,\mu} f(z)\right)'}\right\} > \eta$$

by using (10) in the above equation, we get

$$\mathrm{Re}\left\{\frac{pz^p - \sum_{k=n+p}^{\infty} k\alpha_k A_p^{\gamma,\mu}(k) z^k + \lambda\left[p(p-1)z^p - \sum_{k=n+p}^{\infty} k(k-1)\alpha_k A_p^{\gamma,\mu}(k) z^k\right]}{(1-\lambda)z^p - \sum_{k=n+p}^{\infty}(1-\lambda)\alpha_k A_p^{\gamma,\mu}(k) z^k + \lambda p z^p - \sum_{k=n+p}^{\infty} \lambda k \alpha_k A_p^{\gamma,\mu}(k) z^k}\right\} > \eta \tag{16}$$

on letting $|z| < r \to 1^-$ we obtain

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} \alpha_k A_p^{\gamma,\mu}(k) < 1$$

This proves the theorem.

3. CLOSURE THEOREMS

Theorem2:- Let $b_j \geq 0$ for $j \in \{1,2,3, \dots \dots, m\}$ and $\sum_{j=1}^{m} b_j = 1$ the function $f_j(z) \in N_p(\lambda, \eta, \gamma)$ for j=1,2,3,…..,m where fj(z) is defined as

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_{k,j} z^k \qquad (\alpha_{k,j} \geq 0) \tag{17}$$

then the function H(z) is also in the class $N_p(\lambda, \eta, \gamma)$ where

$$H(z) = \sum_{j=1}^{m} b_j f_j(z) = z^p - \sum_{k=n+p}^{\infty} \left[\sum_{j=1}^{m} b_j \alpha_{k,j}\right] z^k \tag{18}$$

Proof:- For $j \in \{1,2,3, \dots \dots, m\}$, we obtain

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} \alpha_{k,j} A_p^{\gamma,\mu}(k) < 1 \tag{19}$$

hence

$$\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k) \left(\sum_{j=1}^{m} b_j \alpha_{k,j}\right)$$

$$= \sum_{j=1}^{m} b_j \left[\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k) \alpha_{k,j}\right] < \sum_{j=1}^{m} b_j = 1$$

this proves the theorem.

Theorem3:- Let f (z) defined by (1) and g(z) defined by (2) are in the class $N_p(\lambda, \eta, \gamma)$. Then the function

$$h(z) = z^p - \sum_{k=n+p}^{\infty} (\alpha_k^2 + \beta_k^2) z^k$$

is also in the class $N_p(\lambda, \eta, \gamma_1)$ where

$$\gamma_1 \leq \inf_k \left[\frac{(k-p)(k-\eta)[1-\lambda(k-1)]}{2(p-\eta)[1-\lambda(p-1)]} \left(A_p^{\gamma,\mu}(k)\right)^2 - 1\right] \tag{20}$$

Proof:- Since $f, g \in N_p(\lambda, \eta, \gamma)$.

$$\sum_{k=n+p}^{\infty} \left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k)\right]^2 \alpha_k^2$$

$$\leq \left[\sum_{k=n+p}^{\infty} \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k) \alpha_k\right]^2 < 1$$

similarly

$$\sum_{k=n+p}^{\infty} \left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k)\right]^2 \alpha_k^2 < 1$$

therefore

$$\sum_{k=n+p}^{\infty} \frac{1}{2} \left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k)\right]^2 (\alpha_k^2 + \beta_k^2) < 1$$

now we must show that

$$\sum_{k=n+p}^{\infty} \left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma_1,\mu}(k)\right] (\alpha_k^2 + \beta_k^2) < 1$$

this inequality holds if

$$\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma_1,\mu}(k) \leq \frac{1}{2}\left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)\right]^2$$

which is equivalent to

$$A_p^{\gamma_1,\mu}(k) = \frac{(k-\eta)[1-\lambda(k-1)]}{2(p-\eta)[1-\lambda(p-1)]}\left[A_p^{\gamma,\mu}(k)\right]^2$$

since $A_p^{\gamma_1,\mu}(k) \geq \frac{\gamma_1+1}{k-p}$, we obtain

$$\frac{\gamma_1+1}{k-p} \leq \frac{1}{2}\left[\frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)\right]^2$$

this gives required result.

4. RADIUS OF STARLIKENESS AND CONVEXITY

Theorem3:- Let the function $J_p^{\gamma,\mu}f(z)$ defined by (10) then $J_p^{\gamma,\mu}f(z)$ is starlike of order δ, ≤ δ <1 in disc $|z| < R_1$ where

$$R_1 = \inf_k \left\{\frac{(1-\delta)(k-\eta)[1-\lambda(k-1)]}{(k-p+1-\delta)(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)\right\}^{\frac{1}{k-p}} \quad (21)$$

Proof:- It is enough to show that

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)'}{J_p^{\gamma,\mu}f(z)} - p\right| < 1-\delta \; for \; |z| < R_1$$

where R_1 is given by (21). By using (10) we have

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)'}{J_p^{\gamma,\mu}f(z)} - p\right| \leq \frac{\sum_{k=n+p}^{\infty}(k-p)\alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p}}{1-\sum_{k=n+p}^{\infty}\alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p}}$$

thus we have

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)'}{J_p^{\gamma,\mu}f(z)} - p\right| < 1-\delta$$

if

$$\sum_{k=n+p}^{\infty}\frac{(k-p+1-\delta)\alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p}}{(1-\delta)} < 1 \quad (22)$$

but by theorem 1 above equation will be true if

$$\frac{(k-p+1-\delta)}{1-\delta}|z|^{k-p} \leq \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)$$

that is if

$$|z| \leq \left\{\frac{(1-\delta)(k-\eta)[1-\lambda(k-1)]}{(k-p+1-\delta)(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)\right\}^{\frac{1}{k-p}}$$

This proves theorem.

Theorem 4:- Let the function $J_p^{\gamma,\mu}f(z)$ defined by (10) then $J_p^{\gamma,\mu}f(z)$ is convex of order δ, 0 ≤ δ <1 in disc $|z| < R_2$ where

$$R_2 = \inf_k \left\{\frac{p(p-\delta)(k-\eta)[1-\lambda(k-1)]}{k(k-\delta)(p-\eta)[1-\lambda(p-1)]}A_p^{\gamma,\mu}(k)\right\}^{\frac{1}{k-p}} \quad (23)$$

Proof:- It is enough to show that

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)''}{\left(J_p^{\gamma,\mu}f(z)\right)'}\right| \leq 1-\delta \text{ for } |z| < R_2$$

where R_2 is given by (23). By using (10) we have

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)''}{\left(J_p^{\gamma,\mu}f(z)\right)'}\right| \le \frac{\sum_{k=n+p}^{\infty} k(k-1)\alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p} - p(p-1)}{p - \sum_{k=n+p}^{\infty} k\alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p}}$$

thus

$$\left|\frac{z\left(J_p^{\gamma,\mu}f(z)\right)''}{\left(J_p^{\gamma,\mu}f(z)\right)'}\right| \le 1-\delta$$

if

$$\sum_{k=n+p}^{\infty} \frac{k(k-\delta)}{p(p-\delta)} \alpha_k A_p^{\gamma,\mu}(k)|z|^{k-p} \le 1$$

by theorem1 above equation will be true if

$$\frac{k(k-\delta)}{p(p-\delta)}|z|^{k-p} \le \frac{(k-\eta)[1-\lambda(k-1)]}{(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k)$$

that is if

$$|z| \le \left\{\frac{p(p-\delta)(k-\eta)[1-\lambda(k-1)]}{k(k-\delta)(p-\eta)[1-\lambda(p-1)]} A_p^{\gamma,\mu}(k)\right\}^{\frac{1}{k-p}}$$

This proves theorem.

5. APPLICATION TO FRACTIONAL CALCULUS

Theorem 5:- Let $f(z) \in N_p(\lambda, \eta, \gamma), \gamma \ge 0$ then

$$|z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}[1 - A|z|^n] \le \left|D_z^{-\beta} f(z)\right| \le |z|^{p+\beta} \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}[1 + A|z|^n] \qquad (24)$$

where

$$A = \frac{(p+1)_n (p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n \Gamma(n+1)}{(\beta+p+1)_n (n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

and f(z) is analytic function

Proof:- By equation (9) we have

$$\frac{\Gamma(p+\beta+1)}{\Gamma(p+1)} z^{-\beta} D_z^{-\beta} f(z) = z^p - \sum_{k=n+p}^{\infty} \alpha_k M_p(k,\beta) z^k \qquad (25)$$

where

$$M_p(k,\beta) = \frac{\Gamma(\beta+p+1)\Gamma(k+1)}{\Gamma(k+\beta+1)\Gamma(p+1)} \qquad (26)$$

but $A_p^{\gamma,\mu}(k)$ is increasing function of k, and also $M_p(k,\beta)$ is a decreasing function for $k \ge n+p$, thus, we have

$$A_p^{\gamma,\mu}(k) \ge \frac{(\mu+1)_n(\vartheta+2)_n}{(\mu-\gamma+\vartheta+2)_n \Gamma(n+1)} \qquad (27)$$

and

$$M_p(k,\beta) \le \frac{\Gamma(\beta+p+1)\Gamma(n+p+1)}{\Gamma(n+p+\beta+1)\Gamma(p+1)} = \frac{(p+1)_n}{(\beta+p+1)_n} \qquad (28)$$

from (25) we conclude that

$$\left|\frac{\Gamma(p+\beta+1)}{\Gamma(p+1)} z^{-\beta} D_z^{-\beta} f(z)\right| \le |z|^p + \frac{(p+1)_n}{(\beta+p+1)_n}|z|^{n+p} \sum_{k=n+p}^{\infty} \alpha_k$$

$$\left|z^{-\beta} D_z^{-\beta} f(z)\right| \le |z|^p \frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}[1 + A|z|^n] \qquad (29)$$

where

$$A=\frac{(p+1)_n(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n\Gamma(n+1)}{(\beta+p+1)_n(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

thus we get

$$\left|D_z^{-\beta}f(z)\right|\leq|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}[1+A|z|^n] \tag{30}$$

also

$$\left|D_z^{-\beta}f(z)\right|\geq|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p+\beta+1)}[1-A|z|^n] \tag{31}$$

Theorem 6:- Let $f(z)\in N_p(\lambda,\eta,\gamma),\gamma\geq 0$ then

$$|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}[1-B|z|^n]\leq\left|D_z^{\beta}f(z)\right|\leq|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}[1+B|z|^n] \tag{32}$$

where

$$B=\frac{(p+1)_n(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n\Gamma(n+1)}{(p-\beta+1)_n(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

and f(z) is analytic function

Proof:- By using equation (7), we have

$$\frac{\Gamma(p-\beta+1)}{\Gamma(p+1)}z^{\beta}D_z^{\beta}f(z)=z^p-\sum_{k=n+p}^{\infty}\alpha_k B_p(k,\beta)z^k \tag{33}$$

where

$$B_p(k,\beta)=\frac{\Gamma(p-\beta+1)\Gamma(k+1)}{\Gamma(k-\beta+1)\Gamma(p+1)} \tag{34}$$

but $A_p^{\gamma,\mu}(k)$ is increasing function of k, and also $B_p(k,\beta)$ is a decreasing function for $k\geq n+p$, thus, we have

$$A_p^{\gamma,\mu}(k)\geq\frac{(\mu+1)_n(\vartheta+2)_n}{(\mu-\gamma+\vartheta+2)_n\Gamma(n+1)} \tag{35}$$

and

$$B_p(k,\beta)\leq\frac{\Gamma(p-\beta+1)\Gamma(n+p+1)}{\Gamma(n+p-\beta+1)\Gamma(p+1)}=\frac{(p+1)_n}{(p-\beta+1)_n} \tag{36}$$

from (33) we conclude that

$$\left|\frac{\Gamma(p-\beta+1)}{\Gamma(p+1)}z^{\beta}D_z^{\beta}f(z)\right|\leq|z|^p+\frac{(p+1)_n}{(p-\beta+1)_n}|z|^{n+p}\sum_{k=n+p}^{\infty}\alpha_k$$

$$\left|z^{\beta}D_z^{\beta}f(z)\right|\leq|z|^p\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}[1+B|z|^n] \tag{37}$$

where

$$B=\frac{(p+1)_n(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n\Gamma(n+1)}{(p-\beta+1)_n(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

thus we get

$$\left|D_z^{\beta}f(z)\right|\leq|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}[1+B|z|^n] \tag{38}$$

also

$$\left|D_z^{\beta}f(z)\right|\geq|z|^{p+\beta}\frac{\Gamma(p+1)}{\Gamma(p-\beta+1)}[1-B|z|^n] \tag{39}$$

Corollary 1:- letting β = 1 in theorem 5 we obtain

$$\frac{|z|^{p+1}}{(p+1)}[1 - A|z|^n] \leq \left|\int_0^z f(t)dt\right| \leq \frac{|z|^{p+1}}{(p+1)}[1 + A|z|^n]$$

where

$$A = \frac{(p-n+2)(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n\Gamma(n+1)}{(p+2)(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

and f(z) is analytic function

Corrollary 2:- Letting β = 0 in theorem 6 we obtain

$$|z|^p[1 - B|z|^n] \leq |f(z)| \leq |z|^p[1 + B|z|^n]$$

where

$$B = \frac{(p-\eta)[1-\lambda(p-1)](\mu-\gamma+\vartheta+2)_n\Gamma(n+1)}{(n+p-\eta)[1-\lambda(n+p-1)](\mu+1)_n(\vartheta+2)_n}$$

and f(z) is analytic function

6. REFERENCES

[1] H.M.Srivastav, " Distortion inequalities for analytic and univalent functions associated with certain fractional calculus and other linear operators(In Analytic and Geometric Inequalities and Applications eds. T.M.Rassias and H.M.Srivastava), Kluwar Academic Pblishers, Vol. 478,1999, pp 349-374.

[2] H.Silverman, " Univalent functions with negative coefficient" Proc. Amer, Math. Soc., Vol 51,1975, pp 109-116.

[3] S.P. Goyal and R.Goyal, " On a class of multivalent functions defined by generalized Ruscheweyh derivatives involving a general fractional derivative operator" J. Indian Acad. Math., Vol27, No.2, (2005), pp 439-456.

[4] A.Tehranchi and S.R. Kulkarni, "Study of the class of univaleny functions with negative coefficient defined by Ruscheweyh derivative" J. Raj. Acad. Phy. Sci., Vol. 5, No. 1 , 2006, pp 105-118.

[5] W.G.Atshan " Application of fractional calculus operators for a new class of univalent functions with negative coefficient defined by hohlov operator" Math. Slovaca, Vol.60, No.1, 2010, pp 75-82

[6] W.G.Atshan and S.R.Kulkarni, " Application of fractional calculus on certain class of p-vaent functions with negative coefficients defined by Ruscheweyh derivative" Int. J. of Math. Ana, Vol.1, 2007, pp 1089-1101.

[7] H.S.Parihar and Ritu Aggarwal " A class of multivalent functions defined by generalized Ruscheweyh derivatives involing a general fractional derivative operator" Proy. J. of Math. , Vol 33, No. 2, 2014, pp 184-204.